An Improved Approach to Design Linear Parameter-Varying Controllers subject to Uncertain Parameter Measurements ⋆

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Abstract: This paper presents an improved approach for the design of linear parameter-varying controllers subject to uncertain parameter measurements. Specifically, we assume both additive and multiplicative uncertainty on the measured parameter, such that the parameter and its measurement assume values in a nonconvex domain. Closed-loop stability and performance is guaranteed by finding a parameter-dependent (PD) Lyapunov matrix such that a PD linear matrix inequality (LMI) is feasible on this nonconvex domain. We propose to express the nonconvex domain as the image of a polynomial spline, such that the PD LMI is equivalently expressed as a PD LMI on a hyperrectangle. To solve the resulting infinite-dimensional LMI problem, we propose a novel relaxation technique that exploits the properties of B-spline basis functions. An extensive numerical example demonstrates the merits of our approach compared to the state of the art.

Keywords: Linear parameter-varying systems, linear matrix inequalities, multivariable feedback control, convex optimisation, uncertain linear systems, H-infinity control.

1. INTRODUCTION

Linear parameter-varying (LPV) control design has been studied for decades and has proven successful in various realistic engineering applications (Mohammadpour and Scherer, 2012; Hoffmann and Werner, 2015).

Although most LPV control design approaches rely on the assumption that *accurate* parameter measurements are available, non-negligible measurement errors are often present in practice. As a result, these errors cause deteriorated performance or might even destabilize the system (Keel and Bhattacharyya, 1997). Therefore, several approaches addressing this shortcoming have been proposed (Daafouz et al., 2008; Sato, 2010, 2011a, 2015; Lacerda et al., 2016). These approaches assume additive and/or multiplicative uncertainty on the parameter measurement, and are characterized by parameter-dependent (PD) linear matrix inequalities (LMIs). In contrast to LPV controller design approaches that do not consider uncertain measurements, the corresponding PD LMIs should hold on a *nonconvex* parameter domain. In addition, structural constraints should be imposed on the optimization variables to guarantee that the controller solely depends on the measured parameter.

The resulting PD LMI problems are numerically intractable, due to infinite-dimensionality of both the optimization variables and the constraints. The former is relieved by imposing a parameterization on the optimization variables. Subsequently, so-called LMI relaxations provide a finite set of LMIs that is sufficient for the PD LMI on the nonconvex parameter domain. Although Pólya's theorem and sum-of-squares (SOS) decompositions provide well-known approaches to derive LMI relaxations for polynomially PD LMIs (Scherer and Hol, 2006; Oliveira and Peres, 2006) on convex domains, nonconvex domains are hard to account for (Sato, 2015). Moreover, these approaches typically provide a very conservative or no solution for low computational complexities, and thus require a high computational complexity to mitigate conservatism or even obtain a feasible solution.

In this paper, we present improved LMI relaxations for LPV control design subject to parameter measurements with multiplicative and additive uncertainty. These relaxations rely on parameterizing PD LMIs in terms of so-called tensor product B-splines (i.e., specific basis functions for piecewise polynomials), see de Boor (2001) and Schumaker (2007). Since B-splines are positive, positivity (negativity) of the corresponding coefficients implies positivity (negativity) of the PD LMI. As tensor product splines are naturally defined on hyperrectangular domains, we systematically express the nonconvex parameter domain as the image of a polynomial spline, such that the corresponding PD LMIs can be parameterized as polynomial splines (i.e., on a hyperrectangle). The latter extends the recently developed approach presented in Hilhorst et al. (2016). In addition to degree elevation, which is exploited to systematically reduce conservatism using Pólya relaxations, B-splines allow knot insertion as an attractive alternative. In contrast to the conservative approach of

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Sato (2015), which relies on relaxations on the convex hull of a nonconvex domain, our relaxations are capable to directly handle the nonconvex domain.

Numerical experiments demonstrate that the presented B-spline relaxations outperform the approach of Sato (2015), as well as alternative solutions based on SOS or Pólya relaxations, both in terms of conservatism and computational complexity.

The paper is organized as follows. First, we formulate the problem, provide a description of the parameter domain and define polynomial splines in Section 2. Then, our main results are presented in Section 3, followed by numerical validations in Section 4. Finally, the conclusions are provided in Section 5.

Notation The set of nonnegative real numbers (integers) is denoted by $\mathbb{R}_+$ ($\mathbb{N}$), while $\mathbb{R}^n$ ($\mathbb{R}^{m\times n}$) is the set of real vectors (matrices) of dimension n ($m \times n$). The generalized time axis $\mathbb{T}$ equals $\mathbb{R}_+$ in continuous time and $\mathbb{N}$ in discrete time. For a matrix-valued function $X : \mathbb{T} \to \mathbb{R}^{m\times n}$, the operator δ represents the time derivative $\delta X(t) = dX(t)/dt$ in continuous time and the forward time shift $\delta X(t) = X(t+1)$ in discrete time.

2. PROBLEM FORMULATION

We consider the finite-dimensional LPV system

$$P(\alpha) : \begin{cases} \delta x = A(\alpha)x + B_w(\alpha)w + B_u(\alpha)u, & x(0) = 0, \\ z = C_z(\alpha)x + D_{zw}(\alpha)w + D_{zu}(\alpha)u, \\ y = C_y(\alpha)x + D_{yw}(\alpha)w \end{cases} \tag{1}$$

with state $x : \mathbb{T} \to \mathbb{R}^{n_x}$, control input $u : \mathbb{T} \to \mathbb{R}^{n_u}$, exogenous input $w : \mathbb{T} \to \mathbb{R}^{n_w}$, regulated output $z : \mathbb{T} \to \mathbb{R}^{n_z}$, and measured output $y : \mathbb{T} \to \mathbb{R}^{n_y}$. All system matrices are real-valued, bounded, and have a piecewise polynomial dependency on the exogenous multivariate parameter $\alpha = (\alpha_1, \dots, \alpha_N) : \mathbb{T} \to \Lambda$, where Λ is the Cartesian product of N closed and bounded intervals:

$$\Lambda := [\underline{\alpha}_1, \overline{\alpha}_1] \times \cdots \times [\underline{\alpha}_N, \overline{\alpha}_N] \subset \mathbb{R}^N. \tag{2}$$

For technical reasons, we require α to be continuously differentiable in the continuous-time case.

A priori known bounds on the rate of variation of α, defined as $\Delta\alpha := \delta\alpha$ in continuous time and $\Delta\alpha := \delta\alpha - \alpha$ in discrete time, are taken into account. We furthermore assume that the parameter measurement, denoted by $\hat{\alpha}$, is subject to additive and multiplicative uncertainty. Consequently, the admissible set of trajectories of $(\alpha, \hat{\alpha})$ can be expressed in the form

$$\mathcal{T} := \left\{ (\alpha, \hat{\alpha}) : \mathbb{T} \to \mathbb{R}^{2N} \;\middle|\; \begin{matrix} (\alpha(t), \Delta\alpha(t)) \in \Omega, \\ (\alpha(t), \hat{\alpha}(t)) \in \Sigma, \ \forall t \in \mathbb{T} \end{matrix} \right\}, \tag{3}$$

where the sets $\Omega \subset \mathbb{R}^{2N}$ and $\Sigma \subset \mathbb{R}^{2N}$ are further described in Subsection 2.1.

The objective is to derive numerically effective conditions to design a dynamic output feedback LPV controller

$$K(\hat{\alpha}) : \begin{cases} \delta x_c = A_c(\hat{\alpha})x_c + B_c(\hat{\alpha})y, & x_c(0) = 0, \\ u = C_c(\hat{\alpha})x + D_c(\hat{\alpha})y, \end{cases} \tag{4}$$

$x_c : \mathbb{T} \to \mathbb{R}^{n_x}$, that stabilizes the LPV system (1) and satisfies closed-loop performance specifications for all $(\alpha, \hat{\alpha}) \in \mathcal{T}$. The corresponding closed-loop system is expressed as

$$H(\alpha, \hat{\alpha}) : \begin{cases} \delta\tilde{x} = \mathcal{A}(\alpha, \hat{\alpha})\tilde{x} + \mathcal{B}(\alpha, \hat{\alpha})w, \\ z = \mathcal{C}(\alpha, \hat{\alpha})\tilde{x} + \mathcal{D}(\alpha, \hat{\alpha})w, \end{cases} \tag{5}$$

with closed-loop state vector $\tilde{x} := [x' \ x_c']'$. The calculation of the closed-loop state-space matrices is straightforward and omitted for the sake of brevity.

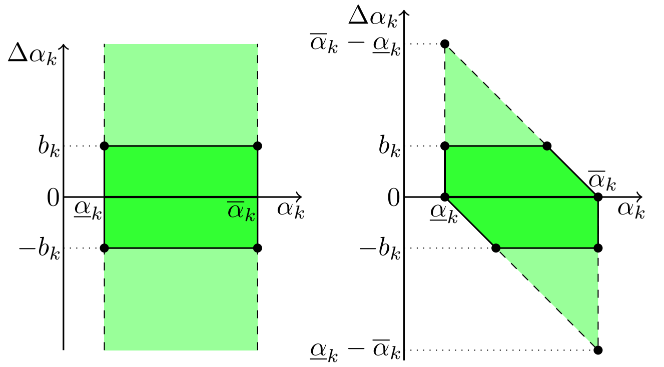

Fig. 1. In the continuous-time case (left), Ω_k equals $[\underline{\alpha}_k, \overline{\alpha}_k] \times \{0\}$ ($b_k = 0$), an unbounded hyperrectangle ($b_k = \infty$) or a bounded hyperrectangle ($0 < b_k < \infty$). In the discrete-time case (right), Ω_k equals $[\underline{\alpha}_k, \overline{\alpha}_k] \times \{0\}$ ($b_k = 0$), a paralellogram ($b_k = \overline{\alpha}_k - \underline{\alpha}_k$) or a hexagon ($0 < b_k < \overline{\alpha}_k - \underline{\alpha}_k$).

2.1 Description of the parameter domain

In this subsection, we provide a detailed description of the parameter domains Ω and Σ in which the time-varying parameter α, its rate of variation $\Delta\alpha$ and the uncertain parameter measurement $\hat{\alpha}$ assume values.

For each parameter α_k, $k = 1, \dots, N$, we consider an a priori known bound $b_k \geq 0$ on its rate of variation:

$$|\Delta\alpha_k(t)| \leq b_k, \quad \forall t \in \mathbb{T}, \quad k = 1, \dots, N.$$

The corresponding region $\Omega_k \subset \mathbb{R}^2$ where $(\alpha_k, \Delta\alpha_k)$ assumes values is different for the continuous-time and the discrete-time case, and furthermore depends on the value of b_k, as shown in Fig. 1. The set Ω is constructed as the Cartesian product of Ω_k, $k = 1, \dots, N$.

Additionally, we assume that the measurement $\hat{\alpha}_k$ is subject to additive and multiplicative uncertainty:

$$\hat{\alpha}_k(t) = (1 + \rho_k(t))(\alpha_k(t) - r_k) + \delta_k(t) + r_k,$$

where $r_k \in [\underline{\alpha}_k, \overline{\alpha}_k]$ is a fixed reference point corresponding to the least uncertain value of α_k, $|\rho_k(t)| \leq \overline{\rho}_k$, and $|\delta_k(t)| \leq \overline{\delta}_k$, for all $t \in \mathbb{T}$, $k = 1, \dots, N$. Note that, for $r_k = 0$, the uncertainty models considered in Sato (2015); Lacerda et al. (2016) are obtained as special cases. The resulting region $\Sigma_k \subset \mathbb{R}^2$ where $(\alpha_k, \hat{\alpha}_k)$ assumes values is shown in Fig. 2, where

$$\begin{aligned} \sigma_{1,k} &:= (1 + \overline{\rho}_k)(\underline{\alpha}_k - r_k) - \overline{\delta}_k + r_k, \\ \sigma_{2,k} &:= (1 - \overline{\rho}_k)(\underline{\alpha}_k - r_k) + \overline{\delta}_k + r_k, \\ \sigma_{3,k} &:= (1 - \overline{\rho}_k)(\overline{\alpha}_k - r_k) - \overline{\delta}_k + r_k, \\ \sigma_{4,k} &:= (1 + \overline{\rho}_k)(\overline{\alpha}_k - r_k) + \overline{\delta}_k + r_k. \end{aligned}$$

The set Σ is constructed as the Cartesian product of Σ_k, $k = 1, \dots, N$.

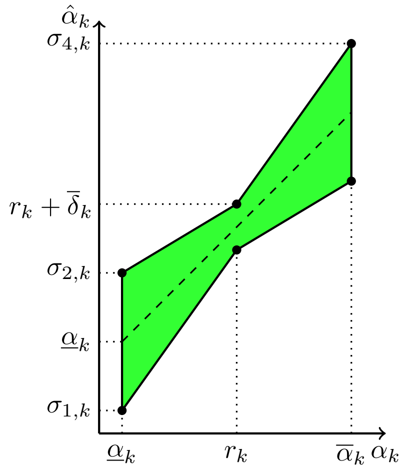

Fig. 2. The set Σ_k is a 6-vertex nonconvex polytope. The dashed line corresponds to $\alpha_k = \hat{\alpha}_k$.

The parameter domain of $(\alpha, \Delta\alpha, \hat{\alpha})$, denoted by $\Gamma \subset \mathbb{R}^{3N}$, can be systematically constructed from the vertices of Ω and Σ. Specifically, by first constructing the 3-dimensional $(\alpha_k, \Delta\alpha_k, \hat{\alpha}_k)$-domains Γ_k, $k = 1, \ldots, N$, we obtain $\Gamma = \Gamma_1 \times \cdots \times \Gamma_N$.

2.2 Polynomial splines

We assume that all the PD system matrices in (1) have a tensor product polynomial spline dependency on α. To facilitate the introduction of tensor product polynomial splines, univariate polynomial splines are defined first. Subsequently, the extension to tensor product polynomial splines is briefly discussed.

Univariate polynomial splines Consider a scalar parameter α on a closed and bounded interval $[\underline{\alpha}, \overline{\alpha}] \subset \mathbb{R}$, and let $\xi = (\xi_0, \ldots, \xi_{l+1})$ be a sequence of points satisfying

$$\underline{\alpha} = \xi_0 < \xi_1 < \cdots < \xi_l < \xi_{l+1} = \overline{\alpha}\,.$$

Then, a matrix $S(\alpha)$ is a polynomial spline (i.e., piecewise polynomial) of degree g with internal break points $\xi_1, \ldots, \xi_l$ and continuity conditions $\nu_1, \ldots, \nu_l$ if there exist polynomial matrices $S_0(\alpha), \ldots, S_l(\alpha)$ of degree g such that

$$\begin{aligned} S(\alpha) &= S_i(\alpha)\,, \quad \text{for } \alpha \in [\xi_i, \xi_{i+1})\,, \quad i = 0, \ldots, l-1\,, \\ S(\alpha) &= S_l(\alpha)\,, \quad \text{for } \alpha \in [\xi_l, \xi_{l+1}]\,, \end{aligned}$$

and

$$\left.\frac{d^{j-1} S_{i-1}}{d\alpha^{j-1}}\right|_{\alpha=\xi_i} = \left.\frac{d^{j-1} S_i}{d\alpha^{j-1}}\right|_{\alpha=\xi_i}, \quad \text{for } \begin{array}{c} j = 1, \ldots, \nu_i\,, \\ i = 1, \ldots, l. \end{array}$$

The latter conditions imply that, in a breakpoint ξ_i, $i \in \{1, \ldots, l\}$, $S(\alpha)$ and its derivatives up to order $(\nu_i - 1)$ are continuous.

By virtue of the Curry-Schoenberg theorem (de Boor, 2001), $S(\alpha)$ can always be expressed in terms of particular normalized B-spline basis functions, by considering the knot sequence

$$\lambda = (\underbrace{\xi_0, \ldots, \xi_0}_{g+1}, \underbrace{\xi_1, \ldots, \xi_1}_{g+1-\nu_1}, \ldots, \underbrace{\xi_l, \ldots, \xi_l}_{g+1-\nu_l}, \underbrace{\xi_{l+1}, \ldots, \xi_{l+1}}_{g+1}).$$

Denoting the i^{th} normalized B-spline basis function of degree g for the knot sequence $\lambda \in \mathbb{R}^{n_\lambda}$ by $B_{i,g,\lambda}(\alpha)$, the PD matrix $S(\alpha)$ is expressed as

$$S(\alpha) = \sum_{i=1}^{n_\lambda - g - 1} C_i\, B_{i,g,\lambda}(\alpha), \tag{6}$$

where C_i, $i = 1, \ldots, n_\lambda - g - 1$ are matrix-valued coefficients.

The main reasons for us to consider B-splines as basis functions for polynomial splines are their positivity and the fact that they can be evaluated in a stable way using the Cox-de Boor recursive formula (de Boor (2001), p. 90).

Tensor product polynomial splines Tensor product polynomial splines constitute a particular multivariate extension of univariate polynomial splines. By specifying a degree g_k and a knot sequence λ_k for every coordinate α_k, $k = 1, \ldots, N$, a tensor product polynomial spline $S(\alpha)$ is defined on Λ in (2) as

$$S(\alpha) = \sum_{i_1=1}^{n_{\lambda_1} - g_1 - 1} \cdots \sum_{i_N=1}^{n_{\lambda_N} - g_N - 1} C_i \underbrace{\left(\prod_{k=1}^{N} B_{i_k, g_k, \lambda_k}(\alpha_k) \right)}_{B_{i,g,\lambda}(\alpha)}, \tag{7}$$

where B_{i_k,g_k,λ_k} and $B_{i,g,\lambda}$ are univariate, respectively, tensor product B-splines. It should be emphasized that the properties of univariate B-splines transfer to tensor product B-splines: they are positive and can be evaluated in a stable way. Also, note that the restriction of a multivariate polynomial to Λ is a tensor product polynomial spline. To simplify terminology, tensor product polynomial splines are named polynomial splines hereafter.

3. MAIN RESULTS

In this section, we present novel LMI relaxations to effectively solve LPV control problems subject to additive and multiplicative uncertainty on the parameter measurement. It is emphasized that we do not present new PD LMI conditions to handle uncertain parameter measurements. Instead, we present a B-spline based approach to derive effective LMI relaxations for recently developed PD LMIs with parameters in nonconvex domains.

3.1 Parameter-dependent LMIs

Generally speaking, approaches for LPV control design subject to uncertainty on the parameter measurement are characterized by sufficient PD LMIs with scalar parameters (Sato, 2010, 2011a, 2015; Lacerda et al., 2016). All these conditions can be expressed as

$$\Upsilon(W, X(\alpha), Y(\alpha, \Delta\alpha), Z(\hat{\alpha}), \varepsilon) \succ 0, \quad \forall (\alpha, \Delta\alpha, \hat{\alpha}) \in \Gamma, \tag{8}$$

where W denotes parameter-independent variables (e.g., a worst-case performance bound or parts of a Lyapunov matrix), $X(\alpha)$ groups system matrices and PD optimization variables, $Y(\alpha, \Delta\alpha)$ denotes optimization variables depending on derivatives/differences (e.g., a Lyapunov matrix), $Z(\hat{\alpha})$ corresponds to variables used for controller reconstruction, and ε is a vector with fixed scalar parameters. As discussed in Subsection 2.1, see also Fig. 2, Γ is nonconvex in the presence of additive and multiplicative uncertainty on the parameter measurement. Moreover, the PD LMI problem (8) is characterized by infinite-dimensional optimization variables and infinitely many constraints, and is thus numerically intractable.

In order to derive B-spline based relaxations (i.e., numerically tractable conditions), the following two conditions should be satisfied:

(1) The PD LMI term in (8) should be parameterized as a polynomial spline.
(2) The associated parameter domain should be a closed and bounded hyperrectangle, since tensor product B-splines are naturally defined on such a domain.

The first requirement is addressed by imposing a polynomial spline parameterization on $X(\alpha)$ and a polynomial parameterization on $Y(\alpha, \Delta\alpha)$ and $Z(\hat{\alpha})$. This results in a finite number of optimization variables. To address the second requirement, we express $\Delta\alpha$ and $\hat{\alpha}$ as the image of polynomial splines. The latter yields a PD LMI term with a polynomial spline dependency on a parameter in a hyperrectangle, as explained next. Subsequently, B-spline based relaxations are presented in Subsection 3.3.

Remark 1. Γ is unbounded in the continuous-time case in the presence of parameters with an unbounded rate of variation. To circumvent the latter issue, we take the Lyapunov matrix independent of the parameters with an unbounded rate of variation (i.e., quadratic stability), see Montagner et al. (2007) and references therein. In contrast to the continuous-time case, Γ is always bounded in discrete time.

3.2 Polynomial spline mappings

The approach proposed in this section transforms the PD LMI (8), which is defined on the nonconvex domain Γ, to an equivalent PD LMI defined on a closed and bounded hyperrectangle. This enables us to parameterize the PD LMI term in (8) as a polynomial spline, which in turn allows the application of B-spline relaxations.

Consider the nonconvex $(\alpha, \hat{\alpha})$-domain Σ defined in Subsection 2.1. We express each parameter $\hat{\alpha}_k \in [\sigma_{1,k}, \sigma_{4,k}]$, see Fig. 2, as the image of a polynomial spline T_k : $[\underline{\alpha}_k, \overline{\alpha}_k] \times [\sigma_{1,k}, \sigma_{4,k}] \to [\sigma_{1,k}, \sigma_{4,k}]$. The latter is illustrated in Fig. 3. The mapping T_k is explicitly given by

$$\hat{\alpha}_k = T_k(\alpha_k, \hat{\beta}_k) = \sum_{i=1}^{3} \sum_{j=1}^{2} C_{ij} B_{i,1,\lambda_{\alpha_k}}(\alpha_k) B_{j,1,\lambda_{\hat{\beta}_k}}(\hat{\beta}_k), \tag{9}$$

with coefficients

$$C_{11} = \sigma_{1,k}, \quad C_{21} = r_k - \overline{\delta}_k, \quad C_{31} = \sigma_{3,k},$$

$$C_{12} = \sigma_{2,k}, \quad C_{22} = r_k + \overline{\delta}_k, \quad C_{32} = \sigma_{4,k},$$

and B-spline basis functions defined by the knot sequences

$$\lambda_{\alpha_k} = (\underline{\alpha}_k, \underline{\alpha}_k, r_k, \overline{\alpha}_k, \overline{\alpha}_k), \quad \lambda_{\hat{\beta}_k} = (\sigma_{1,k}, \sigma_{1,k}, \sigma_{4,k}, \sigma_{4,k}).$$

Subsequently, we define the multivariate mapping

$$T(\alpha, \hat{\beta}) := \Big(T_1(\alpha_1, \hat{\beta}_1), \dots, T_N(\alpha_N, \hat{\beta}_N)\Big) = \hat{\alpha}.$$

In a similar fashion, $\Delta\alpha_k$ is expressed as the image of a polynomial spline $S : [\underline{\alpha}_k, \overline{\alpha}_k] \times [-b_k, b_k] \to [-b_k, b_k]$, $k = 1, \dots, N$, such that $S(\alpha, \Delta\beta) = \Delta\alpha$.

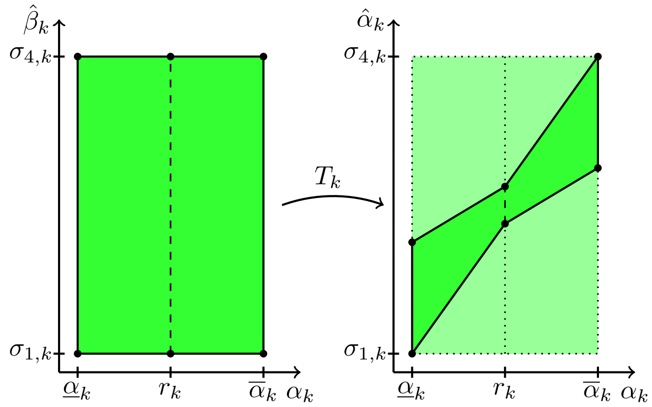

Fig. 3. The polynomial spline mapping T_k, defined in (9), transforms the hyperrectangle $[\underline{\alpha}_k, \overline{\alpha}_k] \times [\sigma_{1k}, \sigma_{4k}]$ to the original parameter domain Σ_k shown in Fig. 2.

It is important to stress that the mappings S and T are defined such that the original parameter α is not transformed. Namely, this enables us to equivalently express the PD LMI (8) as follows:

$$\Upsilon(W, X(\alpha), Y(\alpha, S(\alpha, \Delta\beta)), Z(T(\alpha, \hat{\beta})), \varepsilon) \succ 0, \tag{10}$$

for all $(\alpha, \Delta\beta, \hat{\beta}) \in \Pi$, where

$$\Pi = \Lambda \times [-b_1, b_1] \times \cdots \times [-b_N, b_N] \times [\sigma_{1,1}, \sigma_{4,1}] \times \cdots \times [\sigma_{1,N}, \sigma_{4,N}].$$

The major advantage of the above polynomial spline mappings is the fact that LMI relaxations can be derived on the original (nonconvex) parameter domain. This contrasts approaches that rely on convex outer approximations (see, for instance, Sato (2015)), which require an unnecessary large parameter domain and are thus conservative.

Remark 2. The nonconvex parameter domains considered in Sato (2015) and Lacerda et al. (2016) can also be handled by our approach. In order to not transform the original parameter α, this typically requires a representation of the parameter domain (i.e., the $(\alpha, \Delta\alpha)$-domain and the $(\alpha, \hat{\alpha})$-domain) by a nonminimal set of vertices.

Although the PD optimization problem (10) is defined on a hyperrectangle, it is still numerically intractable due to infinitely many constraints. To tackle this problem, an efficient and elegant solution exploiting polynomial spline parameterizations is presented next.

3.3 B-spline based LMI relaxations

In this subsection, we present a B-spline based approach to derive a numerically tractable (i.e., finite) set of LMIs which, when feasible, guarantees feasibility of the PD LMI (10) for all $(\alpha, \Delta\beta, \hat{\beta}) \in \Pi$. This approach relies on the hyperrectangular structure of the parameter domain Π as well as the positivity of B-splines.

By expressing the left hand side of (8) as a polynomial spline (7) defined on Π, the positivity property of B-splines reveals that

$$C_{i_1,\dots,i_{2N}} \succ 0, \quad i_k = 1, \dots, n_{\lambda_k} - g_k - 1, \quad k = 1, \dots, 2N,$$
$$\Rightarrow \quad \Upsilon(W, X(\alpha), Y(\alpha, S(\alpha, \Delta\beta)), Z(T(\alpha, \hat{\beta})), \varepsilon) \succ 0,$$

for all $(\alpha, \Delta\beta, \hat{\beta}) \in \Pi$. In turn, the PD LMI (8) is feasible for all $(\alpha, \Delta\alpha, \hat{\alpha}) \in \Gamma$. Hence, the relaxation step amounts to imposing positivity on all the B-spline coefficients of the PD LMI term. Generally speaking, imposing positive (negative) definiteness on all the B-spline coefficients of a polynomial spline is sufficient for positive (negative) definiteness of the polynomial spline itself.

Although a finite set of sufficient conditions for the PD LMI (10) is readily derived from the associated B-spline coefficients, a less conservative but larger set of sufficient LMIs is obtained by extending the B-spline basis of $\Upsilon(W, X(\alpha), Y(\alpha, S(\alpha, \Delta\beta)), Z(T(\alpha, \hat{\beta})), \varepsilon)$. Two ways to do this are degree elevation and knot insertion. Further reductions of conservatism can be achieved by increasing the polynomial degree of the PD optimization variables, or by a proper extension of their knot sequence (e.g., equidistant spacing in each coordinate). For details on this, see Van Loock et al. (2016b) and Hilhorst et al. (2016).

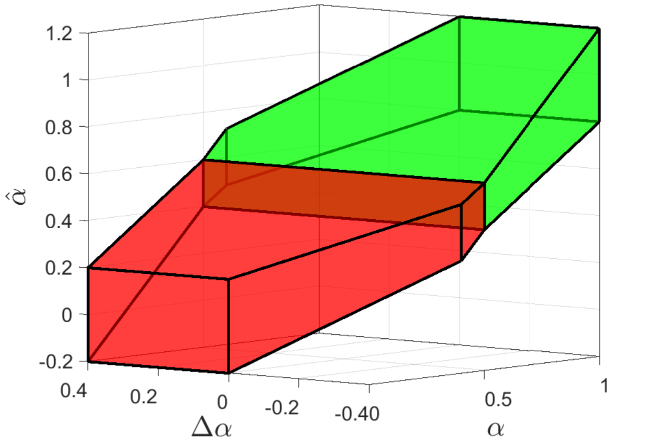

Fig. 4. The nonconvex parameter domain Γ and its subdivision in two convex polytopes.

4. NUMERICAL VALIDATIONS

In this section, numerical comparisons with existing approaches demonstrate the merits of the presented B-spline parameterizations and relaxations for the design of an LPV controller subject to uncertain parameter measurements. The PD LMIs are implemented using the freely available MATLAB software toolbox developed by Van Loock et al. (2016a), which is based on Yalmip (Löfberg (2004)) and facilitates the implementation of optimization problems involving polynomial splines. The proposed B-spline relaxations are performed behind the scenes. We use MOSEK ApS (2015) to solve the LMIs. The used hardware is an Intel Core i5 2.7GHz laptop with 6GB RAM.

We consider a discrete-time LPV model (1) with

$$A(\alpha) = \mu \begin{bmatrix} 1-\alpha & 0 & -2+\alpha \\ 2-\alpha & -1 & 1-\alpha \\ -1+\alpha & 1-3\alpha & -\alpha \end{bmatrix}, \quad B_w(\alpha) = \begin{bmatrix} 0 \\ 1-\alpha \\ \alpha \end{bmatrix},$$

$$B_u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad C_z = [1 \ 1 \ 1], \quad C_y = [1 \ 0 \ 0]$$

$D_{zw} = D_{zu} = 0$, $D_{yw} = 1$ and $\mu = 0.4525$, see Sato (2011b). This model depends affinely on the univariate time-varying parameter $\alpha : \mathbb{N} \to [0, 1]$. The rate of variation of α is bounded as $|\Delta\alpha(t)| \leq 0.4$, and the measured parameter is subject to additive and multiplicative uncertainty. That is,

$$\hat{\alpha}(t) = (1 + \rho(t))(\alpha(t) - 0.5) + \delta(t) + 0.5,$$

where $|\delta(t)| \leq 0.1$ and $|\rho(t)| \leq 0.2$, for all $t \in \mathbb{N}$. Fig. 4 shows the corresponding nonconvex parameter domain Γ.

The objective is to design an unstructured (i.e., full-order) LPV controller (4) with a guaranteed closed-loop $\mathcal{H}_\infty$ performance. Therefore, we start from the recently developed PD LMI with a scalar parameter ε presented in Theorem 1 of Sato (2011b): we select $\varepsilon = 0.1$ and consider LMI formulation (b). It is worth mentioning that this PD LMI is of the form (8). All the PD optimization variables are parameterized as polynomials of degree 2.

The following relaxation approaches are compared:

(1) Pólya relaxations with d degree elevations on the convex hull of Γ (Oliveira and Peres, 2009).
(2) Pólya relaxations with d degree elevations on Γ. This is achieved by subdividing Γ into two convex polytopes (corresponding to $0 \leq \alpha \leq 0.5$, respectively, $0.5 \leq \alpha \leq 1$) and deriving Pólya relaxations for each of the two convex polytopes separately.
(3) B-spline relaxations with d degree elevations on the convex hull of Γ. This yields a paralellogram in $(\alpha, \hat{\alpha})$, which is transformed to a hyperrectangle with an affine transformation.
(4) B-spline relaxations with d degree elevations on Γ.
(5) Same as (3), but with m midpoint refinements instead of d degree elevations.
(6) Same as (4), but with m midpoint refinements instead of d degree elevations.

Specifically, $d = 0, 1, 2$ degree elevations / $m = 0, 1, 2$ midpoint refinements are applied to each coordinate basis. A so-called midpoint refinement corresponds to the addition of a knot in the middle of each breakpoint interval.

Fig. 5 shows the obtained $\mathcal{H}_\infty$ bound versus the computation (i.e., solver) time (in seconds) for each relaxation approach. Compared to the approach in Sato (2015), which relies on relaxations on the convex hull of Γ, significant reductions of conservatism (i.e., 10% to 20% for fixed solver times) are obtained by deriving relaxations on the original nonconvex parameter domain Γ. Additionally, both on the parameter domain Γ and on its convex hull, it is clear that the proposed B-spline based relaxations outperform Pólya relaxations (Oliveira and Peres, 2009), since lower performance bounds are obtained at a lower numerical complexity. Finally, note that midpoint refinements yield a slightly better tradeoff than degree elevations.

Remark 3. We have also implemented sum-of-squares relaxations (Scherer and Hol, 2006) with SOSTOOLS (Papachristodoulou et al., 2013) by using a combination of affine and quadratic constraints to model the parameter domain Γ as well as its convex hull. However, this is no competitive alternative with respect to conservatism versus computation time, as compared to the above approaches.

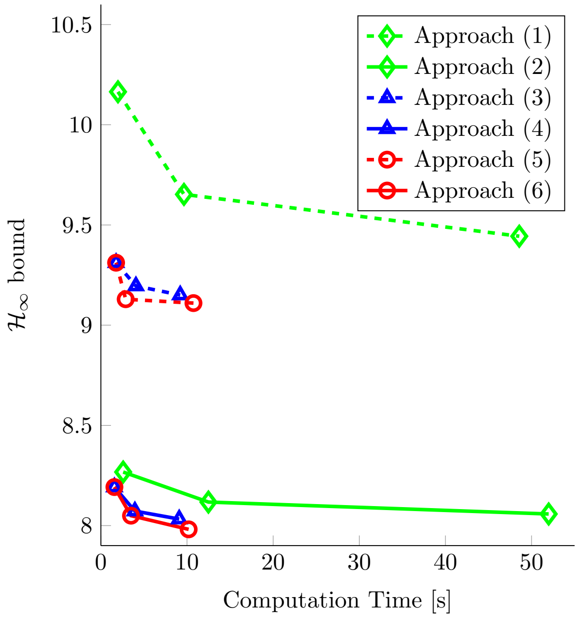

Fig. 5. The $\mathcal{H}_\infty$ bound versus the computation time (in seconds) for the six different relaxation approaches. B-spline relaxations with midpoint refinements on the nonconvex domain provide the best results.

5. CONCLUSIONS

This paper has presented novel LMI relaxations for LPV control design subject to additive and multiplicative uncertainty on the parameter measurement. The corresponding nonconvex parameter domain was expressed as the image of a polynomial spline, such that B-spline based relaxations could be applied to derive a numerically tractable set of LMIs. Numerical experiments have demonstrated that the presented relaxations outperform well-known and widely used approaches both in terms of conservatism and numerical complexity.

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